Logic

Logic *is* a discipline that studies the principles and methods used in correct reasoning

It includes:

- A formal language for expressing statements.
- An inference mechanism (a collection of rules) to reason about valid arguments.

Logic

Logic is a discipline that studies the principles and methods used to construct valid arguments.

- An ***argument*** is a related sequence of statements to demonstrate the truth of an assertion
  - *premises* are assumed to be true
  - *conclusion*, the last statement of the sequence, is taken to be true based on the truth of the other statements.
- An ***argument*** is valid if the conclusion follows logically from the truth of the premises.
- ***Logic*** is the foundation for expressing formal proofs.

Propositional Logic

Propositional Logic is the logic of compound statements built from simpler statements using *Logical Boolean connectives.*

Some applications

- Design of digital electronic circuits.
- Expressing conditions in programs.
- Queries to databases & search engines.

Propositional Logic

A ***proposition*** is a declarative sentence that is either TRUE or FALSE (not both).

Examples:

- The Earth is flat
- 3 + 2 = 5
- I am older than my mother
- Tallahassee is the capital of Florida
- 5 + 3 = 9
- Athens is the capital of Georgia

Propositional Logic

- Letters are used to denote propositions.
  - The most frequently used letters are p, q, r, s and t.
- Example:

p: Grass is green.

- **The truth value** of a proposition is true, denoted by **T**, if it is true, and false, denoted by **F** it is false

Propositional Logic

- **Compound propositions** : built up from simpler propositions using logical operators
  - Frequently corresponds with compound English sentences.

Example:

Given

p: Jack is older than Jill

q: Jill is female

We can build up

r: Jack is older than Jill and Jill is female $(p \wedge q)$

s: Jack is older than Jill or Jill is female $(p \vee q)$

t: Jack is older than Jill and it is not the case that Jill is female $(p \wedge \neg q)$

Propositional Logic - negation

Let p be a proposition.
The *negation* of p is written $\neg p$ and has meaning:

"It is not the case that p."

Truth table for negation:

p	$\neg p$
T	F
F	T

Propositional Logic - conjunction

- *Conjunction operator* "$\wedge$" (*AND*):
  - corresponds to English "and."
  - is a ***binary*** operator in that it operates on two propositions when creating a compound proposition
- Def. Let *p* and *q* be two arbitrary propositions, the conjunction of *p* and *q*, denoted

$$p \wedge q,$$

is true if both *p* and *q* are *true* and *false* otherwise.

Propositional Logic - conjunction

Conjunction operator

$p \wedge q$ *is true* when p and q are both true.

Truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Propositional Logic - disjunction

Disjunction operator $\vee$ (or):

- loosely corresponds to English "or."
- binary operator

Def.: Let *p* and *q* be two arbitrary propositions, the disjunction of *p* and *q*, denoted

$$p \vee q$$

is false when both *p* and *q* are false and true otherwise.

- $\vee$ *is* also called ***inclusive or***
  - *Observe that* $p \vee q$ is true when *p* is true, or ***q*** is true, or both *p* and *q* are true.

Propositional Logic - disjunction

Disjunction operator

$p \lor q$ is true when *p* or *q* (or both) is true.

Truth table for conjunction:

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Propositional Logic - XOR

Exclusive Or operator (⊕):

- corresponds to English "either…or…" (exclusive form of or)
- binary operator

Def.: Let *p* and *q* be two arbitrary propositions, the exclusive or of *p* and *q*, denoted

$$p \oplus q$$

is true when either *p* or *q* (<u>but not both</u>) is true.

Propositional Logic - XOR

Exclusive Or:

$p \oplus q$ is true when p or q (not both) is true.

Truth table for exclusive or:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Propositional Logic- Implication

Implication operator (→):

- binary operator
- similar to the English usage of "if…then…", "implies", and many other English phrases

Def.: Let p and q be two arbitrary propositions, the implication $p \to q$ is false when p is false and q is true, and true otherwise.

$p \to q$ *is true when p is true and q is true,* q is true, or p is false.

$p \to q$ *is f*alse when p is true and q is false.

Example:

r : "The dog is barking."

s : "The dog is awake."

$r \to s$: "If the dog is barking then the dog is awake."

Propositional Logic- Implication

Truth table for implication:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- If the temperature is below 10° F, then water freezes.

Propositional Logic- Biconditional

Biconditional operator $(\leftrightarrow)$:

- Binary operator
- Partly similar to the English usage of "If and only if

Def.: Let *p* and *q* be two arbitrary propositions.

The *biconditional* $p \leftrightarrow q$ *is true* when *q* and *p* have the same truth values and false otherwise.

Example:

p : "The dog plays fetch."

q : "The dog is outside."

$p \leftrightarrow q$: "The plays fetch if and only if it is outside."

Propositional Logic- Biconditional

Truth table for biconditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Propositional Equivalences

- A ***tautology*** is a proposition that is always true.
  - Ex.: $p \vee \neg p$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

- A ***contradiction*** is a proposition that is always false.
  - Ex.: $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

- A ***contingency*** is a proposition that is neither a tautology nor a contradiction.
  - Ex.: $p \rightarrow \neg p$

p	$\neg p$	$p \rightarrow \neg p$
T	F	F
F	T	T

Propositional Logic: Logical Equivalence

- If p and q are propositions, then **p is logically equivalent to** q if their truth tables are the same.
  - "p is equivalent to q." is denoted by $p \equiv q$
- p, q are ***logically equivalent*** if their biconditional $p \leftrightarrow q$ is a tautology.

Propositional Logic: Logical Equivalence

- $p \equiv \neg\neg p$

P	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

The equivalence holds since these two columns are the same.

Propositional Logic: Logical Equivalence

- $p \to q \equiv \neg p \vee q$?

- Does $(p \to q) \leftrightarrow (\neg p \vee q)$?

- Does $(p \to q) \to (\neg p \vee q)$ and $(\neg p \vee q) \to (p \to q)$?

Propositional Logic: Logical Equivalences

- ***Identity***

$$p \wedge \mathbf{T} \equiv p$$
$$p \vee \mathbf{F} \equiv p$$

- ***Domination***

$$p \vee \mathbf{T} \equiv \mathbf{T}$$
$$p \wedge \mathbf{F} \equiv \mathbf{F}$$

- ***Idempotent***

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

- ***Double negation***

$$\neg(\neg p) \equiv p$$

Propositional Logic: Logical Equivalences

- ***Commutative:***

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

- ***Associative:***

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Propositional Logic: Logical Equivalences

- ***Distributive***:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

- ***De Morgan's***:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$ *(De Morgan's I)*

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$ *(De Morgan's II)*

Propositional Logic: Logical Equivalences

- ***Absorption:***

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

- ***Negation:***

$$p \wedge \neg p \equiv \mathbf{F}$$

$$p \vee \neg p \equiv \mathbf{T}$$

- **A useful LE involving $\rightarrow$:**

$$p \rightarrow q \equiv \neg p \vee q$$

Predicate Logic

Define:

UGA(x) = “x is a UGA student.”

Universe of Discourse – *all people*

x is a variable that represents an arbitrary individual in the Universe of Discourse

A **predicate *P***, or propositional function, is a function that maps objects in the universe of discourse to propositions

- **UGA**(Paris Hilton*)* is a **proposition**.
- **UGA***(x)* is **not a proposition.**

UGA(x) is like an English predicate template

- ___________ is a UGA student

Predicate Logic

- Pos(x): x > 0

  Universe of Discourse (UoD): Integers

  Pos(1): 1>0 = T

  Pos(50): 50>0 = T

  Pos(-10): -10>0 = F

- Female(x): x is female

  UoD: all people

  Female(Maria) = T Female(Edward) = F

Predicate Logic

- A predicate that states a property about one object is called a monadic predicate.
  - UGA-Student(x)
  - Parent(x)
  - Female(x)

Predicate Logic

- A predicate of the form $P(x_1,\ldots,x_n)$, n>1 that states the relationships among the objects $x_1,\ldots,x_n$ is called polyadic.

  Also, an n-place predicate or n-ary predicate (a predicate with arity n).

  - It takes n>1 arguments
  - $UoD(x_1,\ldots,x_n) = UoD(x_1) \times UoD(x_2) \times \ldots UoD(x_n)$

- L(x,y): x loves y

  UoD(x) = UoD(y) = all people

  L(Adam,Mary) = T   L(Mike,Jill) = F

- Q(x,y,z): x+y = z

  UoD(x) = UoD(y) = UoD(z): positive integers

  Q(1,1,2) = T   Q(1,2,5) = F

Predicate Logic: Universal Quantifier

Suppose that P(x) is a predicate on some universe of discourse.

The universal quantification of P(x) (∀**x P(x)**) is the **proposition**:

"P(x) is true for all x in the universe of discourse."

∀x P(x) reads "for all x, P(x) is True"

- ∀x P(x) is TRUE means P(x) is true for all x in UoD(x).
- ∀x P(x) is FALSE means there is an x in UoD(x) for which P(x) is false.

Predicate Logic: Universal Quantifier

- ID(x) means x has a student ID.
  UoD(x): UGA students

  $\forall x$ ID(x) means

  every UGA student has a student ID

- IsFemale(x) means x is a Female
  UoD(x): UGA students

  $\forall x$ IsFemale(x) means

  Every UGA student is a female!

Jim is a UGA student who is a male

Jim is called a **counterexample** for $\forall x$ IsFemale(x)

Predicate Logic: Universal Quantifier

In the special case that the universe of discourse U, is finite, ($U = \{a_1, a_2, a_3, ..., a_n\}$)

$$\forall x\ P(x)$$

corresponds to the proposition:

$$P(a_1) \land P(a_2) \land ... \land P(a_n)$$

We can write a program to loop through the elements in the universe and check each for truthfulness.

If all are true, then the proposition is true.

Otherwise it is false!

Predicate Logic: Existential Quantifier

Suppose P(x) is a predicate on some universe of discourse.

The existential quantification of P(x) is the proposition:

"There exists at least one x in the universe of discourse such that P(x) is true."

$\exists$ x P(x) reads "for some x, P(x)" or "There exists x, P(x) is True"

$\exists$x P(x) is **TRUE** means

there is an x in UoD(x) for which P(x) is true.

$\exists$x P(x) is **FALSE** means :

for all x in UoD(x), P(x) is false

Predicate Logic: Existential Quantifier

Examples:

- F(x) means x is female.
  UoD(x): UGA students

  ∃ x F(x) means
  - For some UGA student x, x is a female.
  - There exists a UGA student x who is a female.

- Y(x) means x is less than 13 years old
  UoD(x): UGA students

  ∃ x Y(x) means
  - for some UGA student x, x is less than 13 years old
  - there exists a UGA student x such that x is less than 13 years old

What will be the truth-value of ∃ x Y(x) if the UoD is all people?

Predicate Logic: Existential Quantifier

In the special case that the universe of discourse, U, is finite, (U = $\{x_1, x_2, x_3, \ldots, x_n\}$)

$$\exists x\ P(x)$$

corresponds to the proposition:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

We can write a program to loop through the elements in the universe and check each for truthfulness. If all are false, then the proposition is false. Otherwise, it is true!

Predicates - Quantifier negation

$\exists x\, P(x)$ means "P(x) is true for some x."

What about $\neg\exists x\, P(x)$?

It is not the case that ["P(x) is true for some x."]

"P(x) is not true for all x."

$\forall x\, \neg P(x)$

Existential negation:

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x).$$

Predicates - Quantifier negation

$\forall x\ P(x)$ means "P(x) is true for every x."

What about $\neg\forall x\ P(x)$?

It is not the case that ["P(x) is true for every x."]

"There exists an x for which P(x) is not true."

$$\exists x\ \neg P(x)$$

Universal negation:

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x).$$

Re-Cap

A **predicate *P***, or propositional function, is a function that maps objects in the universe of discourse to propositions

- Predicates can be quantified using the universal quantifier ("for all") $\forall$ or the existential quantifier ("there exists") $\exists$
- Quantified predicates can be negated as follows
  - $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$
  - $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$
- Quantified variables are called "bound"
- Variables that are not quantified are called "free"

Proofs

A *theorem* is a statement that can be proved to be true.

A *proof* is a sequence of statements that form an argument.

Proofs: Inference Rules

- An *Inference Rule:*

premise 1
premise 2 …
∴ *conclusion*

"∴" means "therefore"

Proofs: Modus Ponens

I have a total score over 96.

If I have a total score over 96, then I get an A for the class.

$\therefore$ I get an A for this class

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Proofs: Modus Tollens

If the power supply fails then the lights go out.

The lights are on.

$\therefore$ The power supply has not failed.

$$\frac{\begin{array}{c}\neg q \\ p \rightarrow q\end{array}}{\therefore \neg p}$$

Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Proofs: Addition

I am a student.

∴ I am a student or I am a visitor.

$$\frac{p}{\therefore\ p \lor q}$$

Tautology:

$p \rightarrow (p \lor q)$

Proofs: Simplification

I am a student and I am a soccer player.

∴ I am a student.

$$\frac{p \wedge q}{\therefore\ p}$$

Tautology:

$(p \wedge q) \rightarrow p$

Proofs: Conjunction

I am a student.
I am a soccer player.

$\therefore$ I am a student and I am a soccer player.

$$\frac{\begin{matrix} p \\ q \end{matrix}}{\therefore\ p \wedge q}$$

Tautology:

$((p) \wedge (q)) \rightarrow p \wedge q$

Proofs: Disjunctive Syllogism

I am a student or I am a soccer player.

I am a not soccer player.

∴ I am a student.

$$\begin{array}{l} p \lor q \\ \neg q \\ \hline \therefore p \end{array}$$

Tautology:

$((p \lor q) \land \neg q) \rightarrow p$

Proofs: Hypothetical Syllogism

If I get a total score over 96, I will get an A in the course.

If I get an A in the course, I will have a 4.0 average.

∴ If I get a total score over 96 then I will have a 4.0 average.

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Proofs: Resolution

I am taking CS4540 or I am taking CS4560.

I am not taking CS4540 or I am taking CS7300.

∴ I am taking CS4560 or I am taking CS7300.

$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Tautology:

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

Fallacy of Affirming the Conclusion

If you have the flu then you'll have a sore throat.

You have a sore throat.

∴ You must have the flu.

$$\frac{\begin{array}{l} q \\ p \rightarrow q \end{array}}{\therefore\ p}$$

Fallacy:

$$(q \wedge (p \rightarrow q)) \rightarrow p$$

Abductive reasoning (Useful in real life but not in formal predicate logic)

Fallacy of Denying the Hypothesis

If you have the flu then you'll have a sore throat.

You do not have the flu.

∴ You do not have a sore throat.

$$\begin{array}{l} \neg p \\ p \rightarrow q \\ \hline \therefore \neg q \end{array}$$

Fallacy:

$$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$$

Inference Rules for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore\; P(c)}$$

Universal Instantiation

(for an arbitrary object c from UoD)

$$\frac{P(c)}{\therefore\; \forall x\, P(x)}$$

Universal Generalization

(for any arbitrary element c from UoD)

$$\frac{\exists x\, P(x)}{\therefore\; P(c)}$$

Existential Instantiation

(for some specific object c from UoD)

$$\frac{P(c)}{\therefore\; \exists x\, P(x)}$$

Existential Generalization

(for some object c from UoD)